Géométrie analytique/*Analytic Geometry*

A formula for the Milnor number

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Abstract – We give a formula for the Milnor number of a germ $(X, 0) \subset (\mathbb{C}^{n+1}, 0)$ defined by $f = 0$, $f = f_d + f_{d+k} + \ldots \in \mathbb{C}\{x_0, \ldots, x_n\}$, and such that $\mathrm{Sing}(D) \cap \mathcal{Z}(f_{d+k}) = \emptyset$, where $D = \mathcal{Z}(f_d) \subset \mathbb{P}^n_{\mathbb{C}}$. We prove that the topological type of $(X, 0)$ is determined by the $d+k$-jet of f.

Une formule pour le nombre de Milnor

Résumé – *Dans cette Note, nous donnons une formule pour le calcul du nombre de Milnor d'une singularité isolée* $(X, 0) \subset (\mathbb{C}^{n+1}, 0)$, *définie par une équation* $f = f_d + f_{d+k} + \ldots \in \mathbb{C}\{x_0, \ldots, x_n\}$, f_i *homogène de degré i, et telle que* $\mathrm{Sing}(D) \cap \mathcal{Z}(f_{d+k}) = \emptyset$, *où* $D = \mathcal{Z}(f_d) \subset \mathbb{P}^n_{\mathbb{C}}$. *Nous démontrons que, sous ces hypothèses, le type topologique de* $(X, 0)$, *(ou de f), est déterminé par le jet de f à l'ordre* $d + k$.

Version française abrégée – Soit $f = f_d + f_{d+k} + \ldots \in \mathbb{C}\{x_0, \ldots, x_n\}$, avec $f_i \in \mathbb{C}[x_0, \ldots, x_n]$ homogène de degré i, et soit $(X, 0) = (f^{-1}(0), 0) \subset (\mathbb{C}^{n+1}, 0)$. Pour h polynôme homogène nous noterons par $\mathcal{Z}(h) \subset \mathbb{P}^n_{\mathbb{C}}$ l'ensemble de zéros de h dans $\mathbb{P}^n_{\mathbb{C}}$. Soit $D = \mathcal{Z}(f_d) \subset \mathbb{P}^n_{\mathbb{C}}$ le projectivisé du cône tangent de $(X, 0)$ à l'origine. Nous supposerons tout le long de cette Note que le germe $(X, 0)$ vérifie la condition $\mathrm{Sing}(D) \cap \mathcal{Z}(f_{d+k}) = \emptyset$ $(\star)$. La condition $(\star)$ implique que $\mathrm{Sing}(D)$ est fini et il s'ensuit que $(X, 0)$ est une singularité isolée (*voir* théorème 1).

PROPOSITION 1. – *Soit* $f = f_d + f_{d+k} + \ldots \in \mathbb{C}\{x_0, \ldots, x_n\}$ *tel que* $\mathrm{Sing}(D)$ *est fini. Supposons aussi que* $\mathrm{Sing}(D) \subset \mathbb{P}^n \backslash \mathcal{Z}(x_0)$. *Alors,* $\{f_{x_1}, \ldots, f_{x_n}\}$ *est une base standard de l'idéal* $I = (f_{x_1}, \ldots, f_{x_n})\mathbb{C}\{x_0, \ldots, x_n\}$.

Considérons maintenant la courbe polaire Γ_H de $(X, 0)$ par rapport à un hyperplan générique $H \subset \mathbb{C}^{n+1}$. Par [12], on a $\mathrm{mult}_0(\Gamma_H) = (\Gamma_H, H)_0 = \mu^{(n)}$, où $\mu^{(n)}$ est le nombre de Milnor d'une section hyperplane générale. Puisque $\mathrm{Sing}(D)$ est fini, on a $\mu^{(n)}(X, 0) = (d-1)^n$. Choisissons les coordonnées $x_0, \ldots, x_n$ telles que $H = \mathcal{Z}(x_0)$. Alors, Γ_H est le germe à l'origine défini par les fonctions $f_{x_1}, \ldots, f_{x_n}$ et, d'après la proposition 1, le projectivisé du cône tangent est $C\Gamma_H = \mathcal{Z}(f_{d_{x_1}}, \ldots, f_{d_{x_n}})$.

LEMME 1. – *Si* $P \in C\Gamma_H$, *notons* Γ_P *la réunion des composantes irréductibles de* Γ_H *dont le projectivisé du cône tangent est P. Alors,*

$$\mathrm{mult}_0(\Gamma_P) = \sum_{P \in \cap \mathcal{Z}(f_{d_{x_i}})} \mathrm{long}\left(\frac{\mathcal{O}_{\mathbb{P}^n, P}}{(f_{d_{x_1}}, \ldots, f_{d_{x_n}})}\right)$$

où $i \in \{1, \ldots, n\}$. *En particulier, si* $P \in \mathrm{Sing}(D) \subset C\Gamma_H$, $\mathrm{mult}_0(\Gamma_P) = \mu(D, P) := \mu_P$ *et, par conséquent,*

$$\sum_{P \notin \mathrm{Sing}(D)} \mathrm{mult}_0(\Gamma_P) = (d-1)^n - \sum_{P \in \mathrm{Sing}(D)} \mu_P.$$

Soit $(S, 0)$ le germe à l'origine défini par la fonction f_{x_0}. Alors,

$$(1) \qquad \mu(X, 0) = (d-1)^{n+1} - (d-1) \sum_{P \in \mathrm{Sing}(D)} \mu_P + \sum_{P \in \mathrm{Sing}(D)} (\Gamma_P, S)_0.$$

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Nous obtenons le théorème suivant.

THÉORÈME 2. – *Soit* $(X, 0) \subset (\mathbb{C}^{n+1}, 0)$ *un germe d'hypersurface défini par* $f = f_d + f_{d+k} + \ldots \in \mathbb{C}\{x_0, \ldots, x_n\}$, *vérifiant la condition* $(\star)$. *Alors,* $(X, 0)$ *est une singularité isolée et*

$$\mu(X, 0) = (d-1)^{n+1} + k\Big(\sum_{P \in \mathrm{Sing}(D)} \mu_P\Big).$$

COROLLAIRE 1. – *Supposons que* $(X, 0)$ *et* $f \in \mathbb{C}\{x_0, \ldots, x_n\}$ *vérifient* $(\star)$. *Soit* $g \in \mathbb{C}\{x_0, \ldots, x_n\}$ *un germe tel que* $f \equiv g \bmod \mathfrak{m}^{d+k+1}$. *Alors* $(X, 0)$ *et* $(g^{-1}(0), 0)$ *ont le même type topologique.*

Finalement, dans la proposition 2 nous calculons les invariants polaires $\{e_q/m_q\}_{\mathrm{red}}$ de Teissier pour ces singularités. A la suite de la proposition 2 et des travaux de Siersma [10] et Saito [9] nous donnons une formule pour le polynôme caractéristique de la monodromie et une formule pour le spectre de la singularité $(X, 0)$.

On peut déduire cette formule pour les singularités $f_d + z^{d+k} = 0$ avec $\mathrm{Sing}(D) \cap V(z) = \emptyset$ d'un travail de Iomdine ([6], [7]). La formule dans le cas $k = 1$ est démontrée dans [1], en calculant le polynôme caractéristique de la monodromie. Notre démonstration est purement algébrique et, donc, valable pour tout corps algébriquement clos de caractéristique zéro.

Let $f(x_0, \ldots, x_n) = 0$, $f \in \mathbb{C}\{x_0, \ldots, x_n\}$, be an equation for a germ of hypersurface $(X, 0) \subset (\mathbb{C}^{n+1}, 0)$. Let $f = f_d + f_{d+k} + \ldots$ be the expression of f as a sum of its homogeneous components. Let h be a homogeneous polynomial, we denote by $\mathcal{Z}(h) \subset \mathbb{P}^n$ the projective set of zeros of h. Let $D = \mathcal{Z}(f_d) \subset \mathbb{P}^n$ be the projectivized tangent cone of X at 0. Suppose that the set $\mathrm{Sing}(D) \cap \mathcal{Z}(f_{d+k})$ is empty, then D has only isolated singularities. Moreover, if $n = 2$ then D must be reduced, and if $n > 2$ then D must be irreducible.

THEOREM 1. – *Under the condition* $\mathrm{Sing}(D) \cap \mathcal{Z}(f_{d+k}) = \emptyset$, $(X, 0)$ *defines an isolated singularity of hypersurface in* $\mathbb{C}^{n+1}$.

Proof. – Let $\pi : \tilde{\mathbb{C}}^{n+1} \to (\mathbb{C}^{n+1}, 0)$ be the blowing up of $(\mathbb{C}^{n+1}, 0)$ with center at 0. Let $E \simeq \mathbb{P}^n = \pi^{-1}(0)$ be the exceptional divisor of π and let $\tilde{X}$ be the strict transform of $(X, 0)$. We know that $\mathrm{Sing}(\tilde{X}) \cap E \subsetneqq \mathrm{Sing}(D)$. If $\mathrm{Sing}(D)$ is empty then the germ $(X, 0)$ has an isolated singularity. If that is not the case let $P \in \mathrm{Sing}(D)$ be a singular point, we choose coordinates so that $P = (1 : 0 : \ldots : 0)$ and therefore the equation of f is $f_d(x_0, \ldots, x_n) + x_0^{d+k} + \tilde{f}_{d+k} + \ldots = 0$. So, the local equations of π are $\pi(y_0, \ldots, y_n) = (y_0, y_1 y_0, \ldots, y_n y_0)$ and in affine coordinates P is the origin; that is, the equation of $\tilde{X}$ in a neighbourhood of P is $f_d(1, y_1, \ldots, y_n) + y_0^k(1 + \tilde{f}_{d+k}(1, y_1, \ldots, y_n) + \ldots) = 0$ where $\tilde{f}_{d+k}(1, 0, \ldots, 0) = 0$. We change of local coordinates such that $y_i = z_i$ for all $i \in \{1, \ldots, n\}$ and $z_0 = y_0 w(y_0, y_1, \ldots, y_n)$ where w is a k-th root of $1 + \tilde{f}_{d+k}(1, y_1, \ldots, y_n) + \ldots$ We consider the projection $p : \tilde{X} \to \mathbb{C}^n$ defined by $p(z_0, \ldots, z_n) = (z_1, \ldots, z_n)$. Then the set $\mathrm{Sing}(\tilde{X})$ is contained in the discriminant set of p. But the discriminant set of p is defined by $f_d^{k-1}(1, z_1, \ldots, z_n) = 0$ so that X has an isolated singularity at 0. $\square$

PROPOSITION 1. – *Let* $f(x_0, \ldots, x_n) = 0$, $f \in \mathbb{C}\{x_0, \ldots, x_n\}$, *be an equation for a germ of hypersurface such that* $\mathrm{Sing}(D)$ *is a finite set. If* $\mathrm{Sing}(D) \subset U = \mathbb{P}^n \backslash \mathcal{Z}(x_0)$, *and* $f_{1_{x_1}} \not\equiv 0, \ldots, f_{d_{x_n}} \not\equiv 0$, *then a standard basis for the ideal* $I = (f_{x_1}, \ldots, f_{x_n})\mathbb{C}\{x_0, \ldots, x_n\}$ *is formed by these* n *generators.*

Proof. – In [13], p. 96, it is proved that $\{f_{x_1}, ..., f_{x_n}\}$ is a standard basis for the ideal I if $\{f_{d_{x_1}}, ..., f_{d_{x_n}}\}$ is a regular sequence in the ring $\mathbb{C}[x_0, ..., x_n]$. As $f_{d_{x_i}}$ is a homogeneous polynomial it is enough to prove that $\{f_{d_{x_1}}, ..., f_{d_{x_n}}\}$ is a regular sequence in $\mathbb{C}[x_0, ..., x_n]_{\mathfrak{m}}$ for all maximal ideals $\mathfrak{m}$ of $\mathbb{C}[x_0, ..., x_n]$ with $J := (f_{d_{x_1}}, ..., f_{d_{x_n}}) \subset \mathfrak{m}$, [5], p. 55, [8], p. 98. Suppose that there exists $Y \subset \mathcal{Z}(J)$ such that $\dim Y \geqq 1$. It is clear that $Y \not\subset \mathcal{Z}(x_0)$. Let $P = (P_0 : ... : P_n) \in Y \cap \mathcal{Z}(f_{d_{x_0}}) \subset \mathrm{Sing}(D)$ be a point, of course $P_0 \neq 0$. Let $y_1, ..., y_n$ be the coordinates in the affine open set U, and let $p_i = P_i/P_0$ be the affine coordinates of P. The Krull's dimension $\dim(\mathcal{O}_{Y,P}) \geqq 1$, and so it is equal to the Krull's dimension of

$$A := \frac{\mathbb{C}\{y_1 - p_1, ..., y_n - p_n\}}{(f_{d_{x_1}}(1, y_1, ..., y_n), ..., f_{d_{x_n}}(1, y_1, ..., y_n))}.$$

Then the dimension of A viewed as $\mathbb{C}$-vector space is infinite. But this is impossible because $(p_1, ..., p_n)$ is an isolated singular point of $f_d(1, y_1, ..., y_n) = 0$, use Euler's relation. Then $\dim \mathcal{Z}(J) = 0$ and in this case, the affine cone defined by J in $\mathbb{C}^{n+1}$ has only irreducible components of dimension one. These components are associated with prime ideals $\mathfrak{p}$ of height n. Therefore, for all maximal ideals $\mathfrak{m}$ of $\mathbb{C}[x_0, ..., x_n]$ with $J \subset \mathfrak{m}$, we have that the ideal $J\,\mathbb{C}[x_0, ..., x_n]_{\mathfrak{m}}$ has height n in this regular local ring. So, [5], p. 187, $J\,\mathbb{C}[x_0, ..., x_n]_{\mathfrak{m}}$ is a complete intersection. Therefore, [2], p. 35, $\{f_{d_{x_1}}, ..., f_{d_{x_n}}\}$ is a regular sequence in $\mathbb{C}[x_0, ..., x_n]_{\mathfrak{m}}$. $\square$

Teissier ([11], [12]) and Henry-Merle [4] have proved there exists W_1 a Zariski open set in the Grassmannian of hyperplanes of $(\mathbb{C}^{n+1}, 0)$ such that if $H \in W_1$, the polar curve Γ_H associated to H is a germ of reduced curve that is an isolated complete intersection singularity, transversal to H, and the multiplicity of Γ_H at origin is $m_0(\Gamma_H) = (\Gamma_H, H)_0 = \mu^{(n)}$, where $\mu^{(n)} = \mu^n(X, 0)$ is the Milnor number of the intersection of $(X, 0)$ with a generic hyperplane of $\mathbb{C}^{n+1}$. Moreover, the number of irreducible components of Γ_H is independent of $H \in W_1$.

Suppose that D has a finite number of singular points (otherwise *see* [11], prop. 2.7), then there exists W_2 another Zariski open set in the Grassmannian of hyperplanes of $(\mathbb{C}^{n+1}, 0)$ such that if $H \in W_2$ the projective variety $H \cap D$ is nonsingular. So the topological type of $(X \cap H, 0)$ is the same as the topological type of the germ $(D \cap H, 0)$ and then $\mu^{(i)}(X \cap H, 0) = (d-1)^i$ for all $i \in \{1, ..., n\}$ [11].

Let $H \in W_1 \cap W_2$ be a hyperplane, we choose coordinates so that the equation of H is $\{x_0 = 0\}$. Now, the germ Γ_H is defined by the equations $f_{x_1} = 0, ..., f_{x_n} = 0$, and $\mathrm{Sing}(D) \cap \mathcal{Z}(x_0)$ is a empty set. Let $C\Gamma_H$ be the projectivized tangent cone of Γ_H at the origin; using proposition 1, we have that

$$R := \mathrm{Gr}_m\left(\frac{\mathbb{C}\{x_0, ..., x_n\}}{(f_{x_1}, ..., f_{x_n})}\right) \simeq \frac{\mathbb{C}[x_0, ..., x_n]}{\mathrm{ini}(f_{x_1}, ..., f_{x_n})} \simeq \frac{\mathbb{C}[x_0, ..., x_n]}{(f_{d_{x_1}}, ..., f_{d_{x_n}})}.$$

Therefore $C\Gamma_H$ is the scheme $\mathrm{Proj}(R)$. Let $\{P_1, ..., P_r, P_{r+1}, ..., P_s\}$ be the support of the above scheme, where $P_i \in \mathrm{Sing}(D)$ if $i = 1, ..., r$ and $P_i \notin \mathrm{Sing}(D)$ otherwise. The tangent line direction of each irreducible component of Γ_H can be interpreted as one of the points of $C\Gamma_H$. Let $P \in C\Gamma_H$ be a point and let Γ_P be the union of the irreducible components of Γ_H whose direction of the tangent line at the origin is defined by the point P; $\Gamma_H = \bigcup_{P \in \mathrm{Sing}(D)} \Gamma_P \cup \bigcup_{P \notin \mathrm{Sing}(D)} \Gamma_P$.

LEMMA 1. – *Under the previous conditions, if $P \in C\Gamma_H$, the multiplicity at the origin of Γ_P coincides with the intersection multiplicity at P of the projective hypersurfaces defined* *by the equations* $f_{d_{x_1}} = 0, \ldots, f_{d_{x_n}} = 0$. *Moreover, if* $P \in \mathrm{Sing}(D)$, *this number is the Milnor number of* D *at* P, *which we will denote by* μ_P.

Proof. – By [3], p. 556, the multiplicity at the origin of Γ_H coincides with the multiplicity at the origin of $C\,\Gamma_H$, viewed as an affine cone which in its turn is equal to the degree of the projective variety $C\,\Gamma_H$, [3], p. 543. Let $\mathfrak{m}_P$ be the homogeneous ideal of R associated to the point P, then

$$\mathrm{mult}_0(\Gamma_H) = \sum_{P \in \cap \mathcal{Z}(f_{d_{x_i}})} \mathrm{length}\,(R_{\mathfrak{m}_P}) = \sum_{P \in \cap \mathcal{Z}(f_{d_{x_i}})} \mathrm{length}\left(\frac{\mathcal{O}_{\mathbf{P}^n, P}}{(f_{d_{x_1}}, \ldots, f_{d_{x_n}})}\right)$$

where $i \in \{1, \ldots, n\}$. For Γ_P we apply the same idea to the ring $R_{\mathfrak{m}P}$. When $P \in \mathrm{Sing}(D)$, since $P \notin H$, to calculate the length we use the local equations $f_{d_{x_i}}(1, y_1, \ldots, y_n) = 0$. □

So, we obtain that $\displaystyle\sum_{P \notin \mathrm{Sing}(D)} \mathrm{mult}_0(\Gamma_P) = (d-1)^n - \sum_{P \in \mathrm{Sing}(D)} \mu_P$.

On the other hand, we can consider the germ $(S, 0) \subset (\mathbb{C}^{n+1}, 0)$ given by the equation $f_{x_0} = 0$. Being a hypersurface, its projectivized tangent cone is $\mathcal{Z}(f_{d_{x_0}})$. For calculating the Milnor number of the singularity $(X, 0)$,

$$\mu := \mu(X, 0) = \dim_{\mathbb{C}}\left(\frac{\mathbb{C}\{x_0, \ldots, x_n\}}{(f_{x_0}, \ldots, f_{x_n})}\right) = \mathrm{length}\left(\frac{\mathcal{O}_{\Gamma_H, 0}}{(f_{x_0})}\right) = (\Gamma_H, S)_0,$$

where the last term indicates the intersection multiplicity at the origin of both germs. If $P \notin \mathrm{Sing}(D)$ the intersection of the germs S and Γ_P is transversal, because the intersection of their projectivized tangent cones is empty. In this way, $(\Gamma_P, S)_0$ is the product of the multiplicities of both germs at the origin.

$$(1) \qquad \mu = (\Gamma_H, S)_0 = (d-1)^{n+1} - (d-1) \sum_{P \in \mathrm{Sing}(D)} \mu_P + \sum_{P \in \mathrm{Sing}(D)} (\Gamma_P, S)_0.$$

THEOREM 2. – *Let* $(X, 0)$ *be a germ of isolated singularity of hypersurface defined by* $f^{-1}(0)$, *where* $f : (U, 0) \to (\mathbb{C}, 0)$ *is a germ of holomorphic function in a neighbourhood of the origin* $0 \in U \subset \mathbb{C}^{n+1}$, *and in a way that if we express* f *as*

$$f(x_0, \ldots, x_n) = f_d(x_0, \ldots, x_n) + f_{d+k}(x_0, \ldots, x_n) + \ldots$$

and we call D *the tangent cone, one has that* $\mathrm{Sing}(D) \cap \mathcal{Z}(f_{d+k}) = \varnothing\,(\star)$, *then*

$$\mu(X, 0) = (d-1)^{n+1} + k\,\Big(\sum_{P \in \mathrm{Sing}(D)} \mu_P\Big),$$

where μ_P *denotes the Milnor number of the curve* D *at the singular point* P.

Proof. – The condition we impose on D implies that we can apply the results we have obtained up until now. After (1) we have to calculate $(\Gamma_P, S)_0$ when $P \in \mathrm{Sing}(D)$. We make a projective change of coordinates that leaves the hyperplane H fixed and such that $P = (1 : 0 : \ldots : 0)$. Let $\Gamma_P = \bigcup_{j=1}^{l} \gamma_j$ and let $h_j : (\mathbb{C}, 0) \to (\gamma_j, 0)$, the normalization of γ_j. Since the direction of the tangent line to γ_j at the origin is $(1 : 0 : \ldots : 0)$, one has that $h_j(t) = (t^{a_j}, h_j^1(t), \ldots, h_j^n(t))$ where $\mathrm{ord}_t\, h_j^i(t) > a_j$ for all $i \in \{1, \ldots, n\}$. We know that the hyperplane H is transversal to the polar curve, then

$$\mu_P = \sum_{j=1}^{l} \mathrm{mult}_0(\gamma_j) = \sum_{j=1}^{l} (\gamma_j, z)_0 = \sum_{j=1}^{l} \mathrm{ord}_t(z \circ h_j) = \sum_{j=1}^{l} a_j.$$

We will conclude if we show that, for all $j \in \{1, ..., l\}$, $(\gamma_j, S)_0 = a_j (d+k-1)$, because, substituting into (1) we obtain the required result. But, it is enough to show, for all $j \in \{1, ..., l\}$, that $\operatorname{ord}_t (f_{d_{x_0}} \circ h_j) > a_j (d+k-1)$. If we have proved it, since $P = (1 : 0 : ... : 0) \in \operatorname{Sing}(D)$ then the equation that defines $(X, 0)$ will be like $f_d + x_0^{d+k} + \tilde{f}_{d+k} + ...$ therefore

$$\begin{aligned}(\gamma_j, f_{x_0})_0 &= \operatorname{ord}_t (f_{x_0} \circ h_j) \\ &= \operatorname{ord}_t (f_{d_{x_0}} (h_j) + (d+k) \cdot t^{a_j (d+k-1)} + \tilde{f}_{d+k_{x_0}} (h_j (t)) + ...),\end{aligned}$$

and hence

$$(\gamma_j, f_{x_0})_0 = a_j (d+k-1). \tag{2}$$

As γ_j is one of the components of Γ_H, then when $i \in \{1, ..., n\}$,

$$\operatorname{ord}_t (f_{d_{x_i}} (h_j (t))) = \operatorname{ord}_t ((f_{d+k})_{x_i} (h_j (t)) + ...) \geqq a_j (d+k-1). \tag{3}$$

We use Euler's relation $d \cdot f_d = x_0 \cdot f_{d_{x_0}} + ... + x_n \cdot f_{d_{x_n}}$ to study the order of $f_d \circ h_j$. Composing with h_j and differentiating with respect to t in the Euler's relation we obtain the following equality

$$\begin{aligned}&(d-1) \cdot [f_{d_{x_1}} (h_j (t)) \cdot (h_j^1)' (t) + ... + f_{d_{x_n}} (h_j (t)) \cdot (h_j^n)' (t)] \\ &\quad + \frac{\partial (f_{d_{x_1}} \circ h_j)}{\partial t} \cdot h_j^1 (t) + ... + \frac{\partial (f_{d_{x_n}} \circ h_j)}{\partial t} \cdot h_j^n (t) \\ &= -(d-1) \cdot f_{d_{x_0}} (h_j (t)) \cdot (a_j t^{a_j - 1}) + \frac{\partial (f_{d_{x_0}} \circ h_j)}{\partial t} \cdot (t^{a_j}).\end{aligned}$$

Using (3), one has that the left hand side of the equality has order in t greater than $a_j - 1 + a_j (d+k-1)$. We now study the order in t of the right hand of the equality. Let

$$f_d = \sum_{\alpha_0 + ... + \alpha_n = d} a_{\alpha_0 ... \alpha_n} \cdot x_0^{\alpha_0} \cdot ... \cdot x_n^{\alpha_n}$$

with $\alpha_0 \leqq d-1$, because $P \in D$. Then, $f_{d_{x_0}} (h_j (t)) = t^{a_j (d-1) + m} \cdot w(t)$, where $w(t) \in \mathbb{C}\{t\}$ is a unit and $m \geqq \alpha_1 + ... + \alpha_n = d - \alpha_0 \geqq 1$. Hence we have that

$$\begin{aligned}-(d-1) \cdot f_{d_{x_0}} (h_j (t)) \cdot (a_j t^{a_j - 1}) &= -a_j (d-1) t^{a_j (d-1) + m + a_j - 1} \cdot (w(0) + ...) \\ \frac{\partial (f_{d_{x_n}} \circ h_j)}{\partial t} \cdot (t^{a_j}) &= (a_j (d-1) + m) t^{a_j (d-1) + m + a_j - 1} \cdot (w(0) + ...),\end{aligned}$$

and since $m \geqq 1$, the order in t of the right hand side is $a_j (d-1) + m + a_j - 1$. So that, $a_j - 1 + a_j (d+k-1) < a_j (d-1) + m + a_j - 1 = \operatorname{ord}_t (f_{d_{x_0}} \circ h_j) + a_j - 1$. Hence the formula for the Milnor number has been proved. □

COROLLARY 1. – *Suppose that* $(X, 0)$ *and* $f \in \mathbb{C}\{x_0, ..., x_n\}$ *verify* $(\star)$. *Let* $g \in \mathbb{C}\{x_0, ..., x_n\}$ *be a germ such that* $f \equiv g \bmod \mathfrak{m}^{d+k+1}$, *then the topological types of* $(X, 0)$ *and* $(g^{-1}(0), 0)$ *are the same.*

Teissier [12], defined the polar invariants (e_q, m_q). Using (2) we can calculate the quotients $\{e_q / m_q\}_{\mathrm{red}}$.

Proposition 2. – *Suppose that* $(X, 0)$ *and* $f \in \mathbb{C}\{x_0, ..., x_n\}$ *verify* $(\star)$ *and* $\operatorname{Sing}(D)$ *is not empty.*

(i) *The set* $\{e_q/m_q\}_{\text{red}}$ *is equal to* $\{d+k-1\}$ *if only if* $\operatorname{Sing}(D) = \mathcal{Z}(f_{d_{x_1}}, ..., f_{d_{x_n}})$. *In this case* $(X, 0)$ *is topologically equivalent to the singularity defined by the polynomial* $x_1^d + x_2^d + ... + x_n^d + x_0^{d+k}$.

(ii) *Otherwise, the set* $\{e_q/m_q\}_{\text{red}}$ *is equal to* $\{d-1, d+k-1\}$ *and if* H *is a hyperplane of* $\mathbb{C}^{n+1}$ *such that* $\operatorname{Sing}(D) \cap H$ *is empty,* $H = \mathcal{Z}(l)$ *where* l *is a linear form, then the family* $F(x, t) = f_d(x) + (1-t) f_{d+k}(x) + tl^{d+k}$ *is* μ^**-constant in an open connected set containing* 0 *and* 1. *So,* $(X, 0)$ *has the same topology that the singularity defined by* $F(x, 1)$.

Remark. – The proof is purely algebraic, so it gives effect to the case of algebraically closed field of characteritic zero. The formula the Milnor number can be deduced, for singularities defined by $f_d + x_0^{d+k} = 0$ where $\operatorname{Sing}(D) \cap \mathcal{Z}(x_0) = \emptyset$, from Iomdine ([6], [7]). Artal [1] has calculated the characteristic polynomial of the complex monodromy in the case $k = 1$, obtaining the same formula. Siersma [10], p. 145, has obtained a formula for the characteristic polynomial and Saito [9] has obtained another formula for the spectrum of the singularities used by Iomdine. Then with the proposition 2 we can apply their results to calculate both invariants for the singularities verifying $(\star)$.

Note remise le 12 septembre 1995, acceptée le 19 septembre 1995.

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